Exercise 2

Show that

$$\frac{1}{1/z} = z \qquad (z \neq 0).$$

Solution

The multiplicative inverse of $z = (x, y)$ is given in Equation (6) on page 4.

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \tag{6}$$

As a result,

$$\begin{aligned}
\frac{1}{1/z} &= \frac{1}{z^{-1}} \\
&= (z^{-1})^{-1} \\
&= \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)^{-1} \\
&= \left(\frac{\frac{x}{x^2+y^2}}{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{-y}{x^2+y^2}\right)^2}, \frac{-\left(\frac{-y}{x^2+y^2}\right)}{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{-y}{x^2+y^2}\right)^2} \right) \\
&= \left(\frac{\frac{x}{x^2+y^2}}{\frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}}, \frac{\frac{y}{x^2+y^2}}{\frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}} \right) \\
&= \left(\frac{\frac{x}{x^2+y^2}}{\frac{x^2+y^2}{(x^2+y^2)^2}}, \frac{\frac{y}{x^2+y^2}}{\frac{x^2+y^2}{(x^2+y^2)^2}} \right) \\
&= \left(\frac{\frac{x}{x^2+y^2}}{\frac{1}{x^2+y^2}}, \frac{\frac{y}{x^2+y^2}}{\frac{1}{x^2+y^2}} \right) \\
&= \left(\frac{x}{x^2 + y^2} \times \frac{x^2 + y^2}{1}, \frac{y}{x^2 + y^2} \times \frac{x^2 + y^2}{1} \right) \\
&= (x, y) \\
&= z.
\end{aligned}$$